ON PRODUCTS AND DUALITY OF BINARY, QUADRATIC, REGULAR OPERADS

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Abstract. Since its introduction by Loday in 1995 with motivation from algebraic K-theory, dendriform dialgebras have been studied quite extensively with connections to several areas in mathematics and physics. A few more similar structures have been found recently, such as the tri-, quadri-, ennea- and octo-algebras, with increasing complexity in their constructions and properties. We consider these constructions as operads and their products and duals, in terms of generators and relations, with the goal to clarify and simplify the process of obtaining new algebra structures from known structures and from linear operators.

Keywords: operads, product, duality, dendriform algebra.

1. Introduction

In order to study the periodicity of algebraic K-groups, J.-L. Loday laid out a program in [30] which led him to the concepts of associative dialgebra and dendriform dialgebra [31]. In the next few years, their properties were studied by several authors in areas related to operads [35], homology [17, 18], Hopf algebras [7, 26, 42, 39], combinatorics [16, 37, 2, 3], arithmetic [34] and quantum field theory [16]. See [33] and other articles in the volume for a survey of some of these developments.

Since 2002, quite a few more similar algebra structures have been introduced, such as the associative trialgebra and dendriform trialgebra of Loday and Ronco [38], the dendriform quadri-algebra of Aguiar and Loday [4], the ennea-algebra, the NS-algebra, the dendriform-Nijenhuis algebra and the octo-algebra of Leroux [27, 28, 29]. These algebras have a common property of "splitting associativity", that is, expressing the multiplication of an associative algebra as the sum of a string of binary operations. The operations in the string satisfy a set of relations and the associativity of the multiplication follows from the sum of these relations. The first instance of such algebras, the dendriform dialgebra, has a string of two operators. The later constructions were largely inspired by the connection [1, 10, 27] with Rota-Baxter operators[1] which were introduced by G. Baxter [6] in 1960 and were actively studied in the 1970s [43, 44] and again in recent years in connection with several areas of mathematics and physics [5, 8, 9, 11, 12, 15, 15, 23, 24, 21, 22, 25].

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[1]They used to be called Baxter operators. They are renamed Rota-Baxter operators to distinguish it clearly from the very related Yang-Baxter operators. The latter Baxter is the Australian physicist Rodney Baxter.

Two themes can be found in these recent constructions. One is the construction of a new type of algebras that has the combined features of types of two or more algebras that were previously known. The other is the use of a linear operator with certain features, such as a Rota-Baxter operator, on a known type of algebras to obtain another type of algebras with richer structures. Even though the ideas of the themes are simple, to carry them out for a particular construction can be quite complicated.

The purpose of this paper is to study these constructions in the framework of operads and their products, given by generators and relations. This enables us to clarify, simplify and further generalize the constructions and properties of these recent algebra structures.

Here is a more detailed plan of the paper. In §2, we recall the operads that give rise to the above algebra structures. These operads are binary, quadratic and regular [35] operads with a splitting of associativity. To ease the notation, we call them ABQR operads and the corresponding algebras ABQR algebras. The generator-relation construction of ABQR operads allows a concrete description which is quite simple and can be found in the existing literature [33]. We use this description to formally define the types of such operads.

We then define in §3 products of ABQR algebras that are similar to (but different from) the operad products of Manin-Ginzburg-Kapranov [40, 41, 20, 32]. We show, formalizing the first theme, that some recently obtained ABQR algebras [4, 27, 28, 29] are products of simpler algebras. Properties of products of ABQR algebras are also studied in this section. The subsequent Section 4 considers the dual of a ABQR operad and its relation with the products.

A full understanding of the second theme mentioned above depends on sufficient knowledge of the linear operators that give rise to ABQR algebras. While this topic is being investigated in another project, using the framework introduced here, we can make the second theme precise for most of the operators that we are aware of, when the operators are applied to any types of ABQR algebras. This is presented in §5.

The concept of unit actions of operads has recently been introduced by Loday [35] and used to construct Hopf algebras on the free algebras. In a separate work [13], we investigate the relation between products of operads and their unit actions. See [45] also for a more recent application of products of ABQR operads.

2. ABQR algebras and operads

2.1. **ABQR algebras and their types.** Let $\mathbf{k}$ be a field of characteristic zero. Let A be a vector space over $\mathbf{k}$. The dendriform dialgebra and its generalizations, such as the trialgebra, quadri-algebra, ennea-algebra or Nijenhuis-dendriform algebra, are constructed from a finite set of binary operations

$$\Omega := \{\odot_n : A \otimes_{\mathbf{k}} A \to A,\ n = 1, \cdots, m\}$$

together with a set of "associativity" relations. This construction fits into the general framework of operads [19, 20, 32, 35] that are binary, quadratic and regular. We will first briefly recall these concepts and refer to the above references for details. We then give a more concrete definition of such algebras adapted from [33, 38]. This definition is

easier to work with and applies when the base field $\mathbf{k}$ is replaced by a commutative ring with identity.

An (algebraic) operad is a sequence $\{\mathcal{P}(n)\}$ of finitely generated $\mathbf{k}[S_n]$-modules such that the Schur functor

$$\mathcal{P} : V \mapsto \bigoplus_n (\mathcal{P}(n) \otimes V^{\otimes n})_{S_n}$$

is equipped with a composition law $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ which is associative and unital. An operad $\{\mathcal{P}(n)\}$ is called **binary** if $\mathcal{P}(1) = \mathbf{k}$ and $\mathcal{P}(n), n \geq 3$ are induced from $\mathcal{P}(2)$ by composition; is called **quadratic** if all relations among the binary operations in $\mathcal{P}(2)$ are derived from $\mathcal{P}(3)$; is called **regular** if, moreover, the binary operations have no symmetries (such as $x \cdot y = y \cdot x$), and the induced relations in $\mathcal{P}(3)$ occur in the same order (such as $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, not $(x \cdot y) \cdot z = x \cdot (z \cdot y)$).

By regularity, the space $\mathcal{P}(n)$ is of the form $\mathcal{P}_n \otimes \mathbf{k}[S_n]$ where $\mathcal{P}_n$ is a vector space. So the operad $\{\mathcal{P}(n)\}$ is determined by $\{\mathcal{P}_n\}$. Then a binary, quadratic, regular operad is determined by a pair (Ω, Λ) where $\Omega = \mathcal{P}_2$, called the **space of generators**, and Λ is a subspace of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$, called the **space of relations.** The pair (Ω, Λ) is called the **type** of the operad $\mathcal{P}$, or of the corresponding algebra structure.

A $\mathbf{k}$-vector space A is called a **binary, quadratic, regular algebra of type** (Ω, Λ) if it has binary operations Ω and if, for

$$\Big(\sum_{i=1}^{k} \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_{j=1}^{m} \odot_j^{(3)} \otimes \odot_j^{(4)}\Big) \in \Lambda \subseteq \Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$$

with $\odot_i^{(1)}, \odot_i^{(2)}, \odot_j^{(3)}, \odot_j^{(4)} \in \Omega$, $1 \leq i \leq k$, $1 \leq j \leq m$, we have

$$\sum_{i=1}^{k} (x \odot_i^{(1)} y) \odot_i^{(2)} z = \sum_{j=1}^{m} x \odot_j^{(3)} (y \odot_j^{(4)} z), \forall\, x, y, z \in A. \tag{1}$$

When there is no danger of confusion, we will use Λ to denote the type of an algebra structure.

Since a type (Ω, Λ) is determined by $(\vec{\omega}, \vec{\lambda})$ where $\vec{\omega}$ is a basis of Ω and $\vec{\lambda}$ is a basis of Λ, we also use $(\vec{\omega}, \vec{\lambda})$ to denote for the type of a binary, quadratic, regular operad, as is usually the case in the literature.

We say that a binary, quadratic, regular operad (Ω, Λ) **has a splitting associativity** if there is an element $\star$ of Ω such that $(\star \otimes \star, \star \otimes \star)$ is in Λ [35]. As abbreviation, we call such an operad an **associative BQR operad**, or simply an **ABQR operad**.

Lemma 2.1. *A binary, quadratic, regular operad is ABQR if and only if there is a basis* $\vec{\omega} = \{\omega_i\}$ *of* Ω *such that* $\star = \sum_i \omega_i$ *and there is a basis* $\vec{\lambda} = \{\lambda_j\}_j$ *of* Λ *such that the associativity of* $\star$ *is given by the sum of* λ_j *(splitting associativity):*

$$(\star \otimes \star, \star \otimes \star) = \sum_j \lambda_j.$$

Proof. The if part is clear. For the only if part, given $\star \in \Omega$ such that $(\star \otimes \star, \star \otimes \star)$ is in Λ, complete $\star$ to a basis $\{\star, \omega_2, \cdots, \omega_r\}$ of Ω and then take $\omega_1 = \star - \omega_2 - \cdots - \omega_r$.

Similarly complete $(\star\otimes\star,\star\otimes\star)$ to a basis $\{(\star\otimes\star,\star\otimes\star),\lambda_2,\cdots,\lambda_s\}$ of Λ and then take $\lambda_1=(\star\otimes\star,\star\otimes\star)-\lambda_2-\cdots-\lambda_s$. $\square$

Let (Ω,Λ) and (Ω',Λ') be ABQR operads with associative operations $\star$ and $\star'$ respectively. A **morphism** $f:(\Omega,\Lambda)\to(\Omega',\Lambda')$ is a linear map $\Omega\to\Omega'$ sending $\star$ to $\star'$ and inducing a linear map $\Lambda\to\Lambda'$. An invertible morphism is called an **isomorphism**, and called an **automorphism** if $(\Omega,\Lambda)=(\Omega',\Lambda')$.

2.2. **Examples.** We now describe known examples of dendriform related algebras in the context of operad types that were just defined, as illustrations of the concepts and as preparations for later applications.

1. (Associative algebra) An associative **k**-algebra is a **k**-vector space A with an associative product $\cdot$. This means that it is of type $(\vec{\omega}_A,\vec{\lambda}_A)$ with $\vec{\omega}_A=\{\cdot\}$ and $\vec{\lambda}_A=\{(\cdot\otimes\cdot,\cdot\otimes\cdot)\}$.

2. (Dialgebra) The **dendriform dialgebra** of Loday [33] is defined to be a **k**-vector space D with binary operations $\prec$ and $\succ$ such that

$$(x\prec y)\prec z=x\prec(y\prec z+y\succ z),(x\succ y)\prec z=x\succ(y\prec z),$$
$$(x\prec y+x\succ y)\succ z=x\succ(y\succ z).$$

This means that D is of operad type $(\vec{\omega}_D,\vec{\lambda}_D)$ with $\vec{\omega}_D=\{\prec,\succ\}$ and

$$\vec{\lambda}_D=\{(\prec\otimes\prec,\prec\otimes(\prec+\succ)),(\succ\otimes\prec,\succ\otimes\prec),((\prec+\succ)\otimes\succ,\succ\otimes\succ)\}. \tag{2}$$

The associativity of $\star:=\prec+\succ$ follows from the sum of these equations. If we exchange $\prec$ and $\succ$ in both $\vec{\omega}_D$ and $\vec{\lambda}_D$, we get a type $(\vec{\omega}_D^{op},\vec{\lambda}_D^{op})$ that is isomorphic to $(\vec{\omega}_D,\vec{\lambda}_D)$. It is called the opposite of $(\vec{\omega}_D,\vec{\lambda}_D)$. The same holds for the dendriform trialgebra, and Nijenhuis trialgebra in the following examples.

3. (Trialgebra) The **dendriform trialgebra** of Loday and Ronco [38] is a **k**-vector space T equipped with binary operations $\prec,\succ$ and $\circ$ that satisfy the relations

$$\begin{aligned}&(x\prec y)\prec z=x\prec(y\star z),(x\succ y)\prec z=x\succ(y\prec z),\\&(x\star y)\succ z=x\succ(y\succ z),(x\succ y)\circ z=x\succ(y\circ z),\\&(x\prec y)\circ z=x\circ(y\succ z),(x\circ y)\prec z=x\circ(y\prec z),(x\circ y)\circ z=x\circ(y\circ z)\end{aligned}$$

for $x,y,z\in D$. Here $\star=\prec+\succ+\circ$. This is the ABQR algebra of type $(\vec{\omega}_T,\vec{\lambda}_T)$ with $\vec{\omega}_T=\{\prec,\succ,\circ\}$ and

$$\begin{aligned}\vec{\lambda}_T=\ &\{(\prec\otimes\prec,\prec\otimes\star),(\succ\otimes\prec,\succ\otimes\prec),(\star\otimes\succ,\succ\otimes\succ),(\succ\otimes\circ,\succ\otimes\circ),\\&(\prec\otimes\circ,\circ\otimes\succ),(\circ\otimes\prec,\circ\otimes\prec),(\circ\otimes\circ,\circ\otimes\circ)\}.\end{aligned}\tag{3}$$

Note that the trialgebra contains an associative operation $\circ$ which is part of the splitting of the associative operation $\star$.

4. (NS-algebra) The **NS-algebra** of Leroux [28] is defined with three binary operators $\prec,\succ,\bullet$ that satisfy the relations

$$(x\prec y)\prec z=x\prec(y\star z),(x\succ y)\prec z=x\succ(y\prec z),(x\star y)\succ z=x\succ(y\succ z),$$
$$(x\star y)\bullet z+(x\bullet y)\prec z=x\succ(y\bullet z)+x\bullet(y\star z)$$

for $x, y, z \in D$. Here $\star = \prec + \succ + \bullet$ gives an associative operation. This is the ABQR algebra of type $(\vec{\omega}_N, \vec{\lambda}_N)$ with $\vec{\omega}_N = \{\prec, \succ \ \bullet\}$ and

$$\begin{aligned}\vec{\lambda}_N = \ & \{(\prec \otimes \prec, \prec \otimes \star), (\succ \otimes \prec, \succ \otimes \prec), (\star \otimes \succ, \succ \otimes \succ), \\ & (\star \otimes \bullet + \bullet \otimes \prec, \succ \otimes \bullet + \bullet \otimes \star)\}.\end{aligned} \tag{4}$$

5. (L-dipterous algebras) An L-dipterous algebra [39] is a $\mathbf{k}$-vector space A with two operations $\star$ and $\succ$ such that

$$\vec{\lambda}_L = \{(\star \otimes \star, \star \otimes \star), (\star \otimes \succ, \succ \otimes \succ), (\succ \otimes \star, \succ \otimes \star)\}. \tag{5}$$

An L-anti-dipterous algebra is a vector space A with two operations $\star$ and $\prec$ such that

$$\vec{\lambda}_R = \{(\star \otimes \star, \star \otimes \star), (\prec \otimes \prec, \prec \otimes \star), (\star \otimes \prec, \star \otimes \prec)\}. \tag{6}$$

3. Products of ABQR operads

3.1. **Definitions.** Manin [40, 41] defined two products for quadratic algebras, called the **black circle product** $\bullet$ and **white circle product** $\circ$. They were later generalized for operads [20]. We defined similar products for ABQR operads in terms of the types. The analogue of the black circle product was independently defined by Loday [36] where he called it the **square product** with the notation $\square$. We adopt his terminology and notation. We also call the analogue of the white circle product the **maltese product** with the notation ✠. The justification of the names and notations will be given below. As the referee and Loday pointed out, the square product and maltese product are different from the black circle product and white circle product of Manin-Ginzburg-Kapranov. We will not go further in this direction since the later products will not be needed in this paper. A remark is given at the end of § 4.

Let (Ω_1, Λ_1) and (Ω_2, Λ_2) be the types of two ABQR operads. Define

$$(\Omega_1, \Lambda_1) \square (\Omega_2, \Lambda_2) = (\Omega_1 \otimes \Omega_2, S_{(23)}(\Lambda_1 \otimes \Lambda_2)),$$

$$(\Omega_1, \Lambda_1) ✠ (\Omega_2, \Lambda_2) = (\Omega_1 \otimes \Omega_2, S_{(23)}(\Lambda_1 \otimes \Omega_2^{\otimes 2} + \Omega_1^{\otimes 2} \otimes \Lambda_2)).$$

Here $S_{(23)}$ means the exchange of factor 2 and 3 in the tensor products. We see that relations of the square product (resp. maltese product) take the shape of a square (resp. a (Maltese) cross). We will rephrase these products more precisely for later applications.

For (Ω_1, Λ_1) and (Ω_2, Λ_2) as above, and for $\odot^{(i)} \in \Omega_i, \ i = 1, 2$, we use a column vector $\begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix}$ to denote the tensor product $\odot_1 \otimes \odot_2 \in \Omega_1 \otimes \Omega_2$. The purpose of this is to distinguish it from the tensor product in $\Lambda_i \subseteq \Omega_i^{\otimes 2} \oplus \Omega_i^{\otimes 2}$. Further, for $f_i = (\odot_i^{(1)} \otimes \odot_i^{(2)}, \odot_i^{(3)} \otimes \odot_i^{(4)}) \in \Omega_i^{\otimes 2} \oplus \Omega_i^{\otimes 2}, \ i = 1, 2$, define

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\begin{bmatrix} \odot_1^{(1)} \\ \odot_2^{(1)} \end{bmatrix} \otimes \begin{bmatrix} \odot_1^{(2)} \\ \odot_2^{(2)} \end{bmatrix}, \begin{bmatrix} \odot_1^{(3)} \\ \odot_2^{(3)} \end{bmatrix} \otimes \begin{bmatrix} \odot_1^{(4)} \\ \odot_2^{(4)} \end{bmatrix} \right) \in (\Omega_1 \otimes \Omega_2)^{\otimes 2} \oplus (\Omega_1 \otimes \Omega_2)^{\otimes 2}.$$

This extends by bilinearity to all $f_i \in \Omega_i^{\otimes 2} \oplus \Omega_i^{\otimes 2},\ i=1,2$. More precisely, elements of $\Omega_i^{\otimes 2} \oplus \Omega_i^{\otimes 2}$ are finite sums of the form

$$f_1 = \sum_j (\odot_{1,j}^{(1)} \otimes \odot_{1,j}^{(2)}, \odot_{1,j}^{(3)} \otimes \odot_{1,j}^{(4)}),\ \odot_{1,j}^{(r)} \in \Omega_1,\ r=1,\dots,4$$

and

$$f_2 = \sum_k (\odot_{2,k}^{(1)} \otimes \odot_{2,k}^{(2)}, \odot_{2,k}^{(3)} \otimes \odot_{2,k}^{(4)}),\ \odot_{2,k}^{(r)} \in \Omega_2,\ r=1,\dots,4.$$

We then define

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \sum_{j,k} \left(\begin{bmatrix} \odot_{1,j}^{(1)} \\ \odot_{2,k}^{(1)} \end{bmatrix} \otimes \begin{bmatrix} \odot_{1,j}^{(2)} \\ \odot_{2,k}^{(2)} \end{bmatrix}, \begin{bmatrix} \odot_{1,j}^{(3)} \\ \odot_{2,k}^{(3)} \end{bmatrix} \otimes \begin{bmatrix} \odot_{1,j}^{(4)} \\ \odot_{2,k}^{(4)} \end{bmatrix} \right).$$

We define two subspaces of $(\Omega_1 \otimes \Omega_2)^{\otimes 2} \oplus (\Omega_1 \otimes \Omega_2)^{\otimes 2}$ by

$$\Lambda_1 \,\Box\, \Lambda_2 = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \Big| f_i \in \Lambda_i,\ i=1,2 \right\}, \quad \Lambda_1 \maltese \Lambda_2 = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \Big| f_1 \in \Lambda_1 \text{ or } f_2 \in \Lambda_2 \right\}.$$

So $\Lambda_1 \,\Box\, \Lambda_2$ and $\Lambda_1 \maltese \Lambda_2$ can be regarded as sets of relations for the operator set $\Omega_1 \otimes \Omega_2$.

Definition 3.1. *The* **square product** (*also called* **type product**) *of* (Ω_1, Λ_1) *and* (Ω_2, Λ_2), *denoted by* $(\Omega_1, \Lambda_1) \,\Box\, (\Omega_2, \Lambda_2)$, *is the type* $(\Omega_1 \otimes \Omega_2, \Lambda_1 \,\Box\, \Lambda_2)$. *The* **maltese product** *of* (Ω_1, Λ_1) *and* (Ω_2, Λ_2), *denoted by* $(\Omega_1, \Lambda_1) \maltese (\Omega_2, \Lambda_2)$, *is the type* $(\Omega_1 \otimes \Omega_2, \Lambda_1 \maltese \Lambda_2)$.

Let $\vec{\omega}_i = \{\omega_{i,j}\}$ be a basis of Ω_i, $i=1,2$. Then

$$\vec{\omega}_1 \,\Box\, \vec{\omega}_2 := \left\{ \begin{bmatrix} \omega_{1,j} \\ \omega_{2,k} \end{bmatrix} \right\}_{j,k}$$

is a basis of $\Omega_1 \otimes \Omega_2$. Similarly, let $\vec{\lambda}_i = \{\lambda_{i,j}\}$ be a basis of Λ_i, $i=1,2$. Then

$$\vec{\lambda}_1 \,\Box\, \vec{\lambda}_2 := \left\{ \begin{bmatrix} \lambda_{1,j} \\ \lambda_{2,k} \end{bmatrix} \right\}_{j,k}$$

is a basis of $\Lambda_1 \,\Box\, \Lambda_2$ and takes the shape of a box (matrix). Let $\vec{f}_i$ be a basis of $\Omega_i^{\otimes 2} \oplus \Omega_i^{\otimes 2}$, $i=1,2$. Then a spanning set of $\Lambda_1 \maltese \Lambda_2$ is given by

$$(\vec{\lambda}_1 \,\Box\, \vec{f}_2) \bigcup (\vec{f}_1 \,\Box\, \vec{\lambda}_2).$$

This set is not linearly independent. The union even has overlap when $\vec{f}_i$ is extended from $\vec{\lambda}_i$. Then the union takes the shape of a cross. We will mostly consider the type product $\Box$. Concerning the duality (or the lack of it) between the products $\Box$ and $\maltese$, see §4 for details.

3.2. **Basic properties.**

3.2.1. *Transposes and isomorphisms.* Let $(\Omega_1 \otimes \Omega_2, \Lambda_1 \square \Lambda_2)$ be the type product of two ABQR algebras. We define the **transpose** of $(\Omega_1 \otimes \Omega_2, \Lambda_1 \square \Lambda_2)$ by $(\Omega_2 \otimes \Omega_1, \Lambda_2 \square \Lambda_1)$. It is obtained by changing the two factors in $\begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix}$ throughout $\Omega_1 \otimes \Omega_2$ and $\Lambda_1 \square \Lambda_2$.

Lemma 3.2. *Let (Ω_i, Λ_i), $i = 1, 2$, be the types of two ABQR operads with associative operations $\star_i$.*

1. *$(\Omega_1 \otimes \Omega_2, \Lambda_1 \square \Lambda_2)$ is an ABQR operad with associative operation $\begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix}$.*
2. *$(\Omega_1 \otimes \Omega_2, \Lambda_1 \square \Lambda_2)$ is isomorphic to its transpose $(\Omega_2 \otimes \Omega_1, \Lambda_2 \square \Lambda_1)$.*
3. *If (Ω_1, Λ_1) is isomorphic to (Ω_1', Λ_1') and (Ω_2, Λ_2) is isomorphic to (Ω_2', Λ_2'), then $(\Omega_1 \otimes \Omega_2, \Lambda_1 \square \Lambda_2)$ is isomorphic to $(\Omega_1' \otimes \Omega_2', \Lambda_1' \square \Lambda_2')$.*

Proof. (1). Since $\star_i$ is associative, we have $(\star_i \otimes \star_i, \star_i \otimes \star_i) \in \Lambda_i$. Thus the product $\begin{bmatrix} (\star_1 \otimes \star_1, \star_1 \otimes \star_1) \\ (\star_2 \otimes \star_2, \star_2 \otimes \star_2) \end{bmatrix}$ is in $\Lambda_1 \square \Lambda_2$. But this product is just $\left(\begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix} \otimes \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix}, \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix} \otimes \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix} \right)$. So $\begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix}$ is associative.

(2). The linear map $\Omega_1 \otimes \Omega_2 \to \Omega_2 \otimes \Omega_1$ sending $\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$ to $\begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$ is bijective that sends $\begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix}$ to $\begin{bmatrix} \star_2 \\ \star_1 \end{bmatrix}$ and induces a bijective linear map $\Lambda_1 \otimes \Lambda_2 \to \Lambda_2 \otimes \Lambda_1$. This proves the claim.

The proof of (3) is similar. □

3.2.2. *Tensor products.*

Proposition 3.3. *Let $(D_1, \Omega_1, \Lambda_1)$ and $(D_2, \Omega_2, \Lambda_2)$ be ABQR algebras of type (Ω_1, Λ_1) and (Ω_2, Λ_2) respectively. For $a_1 \otimes a_2, b_1 \otimes b_2 \in D_1 \otimes D_2$ and $\odot_1 \in \Omega_1, \odot_2 \in \Omega_2$, define*

$$(a_1 \otimes a_2) \begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix} (b_1 \otimes b_2) = (a_1 \odot_1 b_1) \otimes (a_2 \odot_2 b_2).$$

This defines an ABQR algebra of type $(\Omega_1 \otimes \Omega_2, \Lambda_1 \square \Lambda_2)$ on $D_1 \otimes D_2$.

For example, when both (Ω_1, Λ_1) and (Ω_2, Λ_2) are of the type of a dendriform dialgebra, then $D_1 \otimes D_2$ is a quadri-algebra. See [4, 1.5]. When both types are of a dendriform trialgebra, then $D_1 \otimes D_2$ is an ennea-algebra. When the two types are of trialgebra and of NS-algebra, respectively, then $D_1 \otimes D_2$ is a dendriform-Nijenhuis algebra [28].

Proof. We only need to verify that the relations in $\Lambda_1 \square \Lambda_2$ are satisfied by $D_1 \otimes D_2$.

Recall that $\Lambda_1 \square \Lambda_2$ consists of elements of the form $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ with $f_i \in \Lambda_i$, $i = 1, 2$. Further, if

$$f_i = \sum_{j_i} \left(\odot_{i,j_i}^{(1)} \otimes \odot_{i,j_i}^{(2)}, \odot_{i,j_i}^{(3)} \otimes \odot_{i,j_i}^{(4)} \right) \in \Lambda_i, \ i = 1, 2,$$

then

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \sum_{j_1, j_2} \left(\begin{bmatrix} \odot_{1,j_1}^{(1)} \\ \odot_{2,j_2}^{(1)} \end{bmatrix} \otimes \begin{bmatrix} \odot_{1,j_1}^{(2)} \\ \odot_{2,j_2}^{(2)} \end{bmatrix}, \begin{bmatrix} \odot_{1,j_1}^{(3)} \\ \odot_{2,j_2}^{(3)} \end{bmatrix} \otimes \begin{bmatrix} \odot_{1,j_1}^{(4)} \\ \odot_{2,j_2}^{(4)} \end{bmatrix} \right).$$

For $x_i, y_i, z_i \in D_i,\ i = 1, 2$, we have

$$\begin{aligned}
&\sum_{j_1,j_2} \left((x_1 \otimes x_2) \begin{bmatrix} \odot^{(1)}_{1,j_1} \\ \odot^{(1)}_{2,j_2} \end{bmatrix} (y_1 \otimes y_2) \right) \begin{bmatrix} \odot^{(2)}_{1,j_1} \\ \odot^{(2)}_{2,j_2} \end{bmatrix} (z_1 \otimes z_2) \\
= &\sum_{j_1,j_2} \Big((x_1 \odot^{(1)}_{1,j_1} y_1) \odot^{(2)}_{1,j_1} z_1 \Big) \otimes \Big((x_2 \odot^{(1)}_{2,j_2} y_2) \odot^{(2)}_{2,j_2} z_2 \Big) \\
= &\left(\sum_{j_1} (x_1 \odot^{(1)}_{1,j_1} y_1) \odot^{(2)}_{1,j_1} z_1 \right) \otimes \left(\sum_{j_2} (x_2 \odot^{(1)}_{2,j_2} y_2) \odot^{(2)}_{2,j_2} z_2 \right) \\
= &\left(\sum_{j_1} x_1 \odot^{(3)}_{1,j_1} (y_1 \odot^{(4)}_{1,j_1} z_1) \right) \otimes \left(\sum_{j_2} x_2 \odot^{(3)}_{2,j_2} (y_2 \odot^{(4)}_{2,j_2} z_2) \right) \\
= &\sum_{j_1,j_2} x \begin{bmatrix} \odot^{(3)}_{1,j_1} \\ \odot^{(3)}_{2,j_2} \end{bmatrix} \left(y \begin{bmatrix} \odot^{(4)}_{1,j_1} \\ \odot^{(4)}_{2,j_2} \end{bmatrix} z \right).
\end{aligned}$$

This is what we want. □

3.3. **Examples.** We now show that some recent generalizations of dendriform dialgebras are products of more basic algebras. Therefore, their generators and relations can be easily described.

3.3.1. *Quadri-algebra.* The quadri-algebra of Aguiar and Loday [4] is defined by four binary operations $\{\nearrow, \nwarrow, \searrow, \swarrow\}$ and 9 relations. Using the auxiliary operations

$$\wedge = \nearrow + \nwarrow, \vee = \searrow + \swarrow, \prec = \nwarrow + \swarrow, \succ = \nearrow + \searrow, \star = \wedge + \vee = \prec + \succ,$$

the 9 relations are given in the following 3×3 matrix.

$$\begin{array}{lll}
(x \nwarrow y) \nwarrow z = x \nwarrow (y \star z) & (x \nearrow y) \nwarrow z = x \nearrow (y \prec z) & (x \wedge y) \nearrow z = x \nearrow (y \succ z) \\
(x \swarrow y) \nwarrow z = x \swarrow (y \wedge z) & (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z) & (x \vee y) \nearrow z = x \searrow (y \nearrow z) \\
(x \prec y) \swarrow z = x \swarrow (y \vee z) & (x \succ y) \swarrow z = x \searrow (y \swarrow z) & (x \star y) \searrow z = x \searrow (y \searrow z)
\end{array} \tag{7}$$

We will put quadri-algebra in the context of type products. Recall that the dendriform dialgebra of Loday is of type $(\vec{\omega}_D, \vec{\lambda}_D)$ with $\vec{\omega}_D = \{\prec, \succ\}$ and

$$\vec{\lambda}_D = \{(\prec \otimes \prec, \prec \otimes \star), (\succ \otimes \prec, \succ \otimes \prec), (\star \otimes \succ, \succ \otimes \succ)\}$$

where $\star = \succ + \prec$.

Proposition 3.4. *The quadri-algebra is isomorphic to the ABQR operad of type*

$$(\Omega_D, \Lambda_D) \,\square\, (\Omega_D, \Lambda_D) = (\Omega_D \otimes \Omega_D, \Lambda_D \,\square\, \Lambda_D).$$

Since the two factors of the product are the same, the transpose of a quadri-algebra is the same algebra. Since the opposite of a dialgebra is still a dialgebra, by Lemma 3.2, if we exchange one or both pairs of $\prec$ and $\succ$, we still obtain a quadri-algebra. The opposite quadri-algebra in [4] is obtained when both pairs are exchanged. As a consequence, applying every element from the dihedral group D_4 of order 8 to a quadri-algebra again gives a quadri-algebra.

Proof. We first define a bijection between the binary operations of the quadri-algebra and the binary operations

$$\vec{\omega}_D \otimes \vec{\omega}_D = \left\{ \begin{bmatrix} \prec \\ \prec \end{bmatrix}, \begin{bmatrix} \prec \\ \succ \end{bmatrix}, \begin{bmatrix} \succ \\ \prec \end{bmatrix}, \begin{bmatrix} \succ \\ \succ \end{bmatrix} \right\}$$

of the type product which is given in the following table.

$$\begin{array}{lll} \nwarrow \leftrightarrow \left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right], & \nearrow \leftrightarrow \left[\begin{smallmatrix} \prec \\ \succ \end{smallmatrix}\right], & \wedge \leftrightarrow \left[\begin{smallmatrix} \prec \\ \star \end{smallmatrix}\right] \\ \swarrow \leftrightarrow \left[\begin{smallmatrix} \succ \\ \prec \end{smallmatrix}\right], & \searrow \leftrightarrow \left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right], & \vee \leftrightarrow \left[\begin{smallmatrix} \succ \\ \star \end{smallmatrix}\right] \\ \prec \leftrightarrow \left[\begin{smallmatrix} \star \\ \prec \end{smallmatrix}\right], & \succ \leftrightarrow \left[\begin{smallmatrix} \star \\ \succ \end{smallmatrix}\right], & \star \leftrightarrow \left[\begin{smallmatrix} \star \\ \star \end{smallmatrix}\right] \end{array} \tag{8}$$

Here the entries on the third row and column are defined to be the sums along the corresponding projections.

To help visualizing this bijection, the reader can imagine the xy plane as sitting in the 3-dimensional coordinate system in the usual way. Thus the plane is laying flat with the x-axis pointing outwards to the reader and the y-axis pointing to the right. Also $\prec$ (resp. $\succ$) points to the negative (resp. positive) direction of the axis. Then, for example, the northwest arrow $\nwarrow$ should be visualized not pointing up and left, but rather inward and left to the third quadrant — the negative direction in both the x and y coordinates. This agrees with the meaning of $\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right]$.

Under the bijection in Eq. (8), the relation matrix (7) is sent to

$$\begin{array}{lll}
(x\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right](y\left[\begin{smallmatrix} \star \\ \star \end{smallmatrix}\right]z), & (x\left[\begin{smallmatrix} \prec \\ \succ \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \prec \\ \succ \end{smallmatrix}\right](y\left[\begin{smallmatrix} \star \\ \prec \end{smallmatrix}\right]z), & (x\left[\begin{smallmatrix} \prec \\ \star \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \prec \\ \succ \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \prec \\ \succ \end{smallmatrix}\right](y\left[\begin{smallmatrix} \star \\ \succ \end{smallmatrix}\right]z), \\
(x\left[\begin{smallmatrix} \succ \\ \prec \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \succ \\ \prec \end{smallmatrix}\right](y\left[\begin{smallmatrix} \prec \\ \star \end{smallmatrix}\right]z), & (x\left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right](y\left[\begin{smallmatrix} \prec \\ \prec \end{smallmatrix}\right]z), & (x\left[\begin{smallmatrix} \succ \\ \star \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \prec \\ \succ \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right](y\left[\begin{smallmatrix} \prec \\ \succ \end{smallmatrix}\right]z), \\
(x\left[\begin{smallmatrix} \star \\ \prec \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \succ \\ \prec \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \succ \\ \prec \end{smallmatrix}\right](y\left[\begin{smallmatrix} \succ \\ \star \end{smallmatrix}\right]z), & (x\left[\begin{smallmatrix} \star \\ \succ \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \succ \\ \prec \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right](y\left[\begin{smallmatrix} \succ \\ \prec \end{smallmatrix}\right]z), & (x\left[\begin{smallmatrix} \star \\ \star \end{smallmatrix}\right]y)\left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right]z = x\left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right](y\left[\begin{smallmatrix} \succ \\ \succ \end{smallmatrix}\right]z)
\end{array}$$

which is simply the matrix of $\vec{\lambda}_D \,\Box\, \vec{\lambda}_D$. $\Box$

3.3.2. *Ennea-algebra.* The ennea-algebra (or 1-ennea-algebra) of Leroux [27] has 9 binary operations

$$\nwarrow, \uparrow, \nearrow, \prec, \circ, \succ, \swarrow, \downarrow, \searrow$$

and 49 relations.

Proposition 3.5. *The ennea-algebra is isomorphic to the type product*

$$(\Omega_T, \Lambda_T) \,\Box\, (\Omega_T, \Lambda_T)$$

where (Ω_T, Λ_T) is the type of the dendriform trialgebra of Loday and Ronco.

Again it is apparent that exchanging $\begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix} \leftrightarrow \begin{bmatrix} \odot_2 \\ \odot_1 \end{bmatrix}$ gives the transpose of the relation matrix. So the transpose of an ennea-algebra is the same algebra. It is also clear that the opposite algebra is the self-product of the opposite trialgebra, and elements in D_4 give algebras isomorphic to the ennea-algebra.

Proof. Recall that the dendriform trialgebra of Loday and Ronco [38] is of type $(\vec{\omega}_T, \vec{\lambda}_T)$ with $\vec{\omega}_T = \{\prec, \succ\ \circ\}$ and

$$\begin{aligned}\vec{\lambda}_T = \ & \{(\prec \otimes \prec, \prec \otimes \star), (\succ \otimes \prec, \succ \otimes \prec), (\star \otimes \succ, \succ \otimes \succ), (\succ \otimes \circ, \succ \otimes \circ),\\ & (\prec \otimes \circ, \circ \otimes \succ), (\circ \otimes \prec, \circ \otimes \prec), (\circ \otimes \circ, \circ \otimes \circ)\}.\end{aligned}$$

Similar to the quadri-algebra, we first give a bijection between the operations of Leroux and those in $\Omega_T \otimes \Omega_T$ in the following table. The entries on the fourth row and column are defined to be the sums along the corresponding projections.

$$\begin{array}{llll}
\nwarrow \leftrightarrow \begin{bmatrix} \prec \\ \prec \end{bmatrix}, & \uparrow \leftrightarrow \begin{bmatrix} \prec \\ \circ \end{bmatrix}, & \nearrow \leftrightarrow \begin{bmatrix} \prec \\ \succ \end{bmatrix}, & \wedge \leftrightarrow \begin{bmatrix} \prec \\ \star \end{bmatrix} \\
\prec \leftrightarrow \begin{bmatrix} \circ \\ \prec \end{bmatrix}, & \circ \leftrightarrow \begin{bmatrix} \circ \\ \circ \end{bmatrix}, & \succ \leftrightarrow \begin{bmatrix} \circ \\ \succ \end{bmatrix}, & \star \leftrightarrow \begin{bmatrix} \circ \\ \star \end{bmatrix} \\
\swarrow \leftrightarrow \begin{bmatrix} \succ \\ \prec \end{bmatrix}, & \downarrow \leftrightarrow \begin{bmatrix} \succ \\ \circ \end{bmatrix}, & \searrow \leftrightarrow \begin{bmatrix} \succ \\ \succ \end{bmatrix}, & \vee \leftrightarrow \begin{bmatrix} \succ \\ \star \end{bmatrix} \\
\triangleleft \leftrightarrow \begin{bmatrix} \star \\ \prec \end{bmatrix}, & \bar{\circ} \leftrightarrow \begin{bmatrix} \star \\ \circ \end{bmatrix}, & \triangleright \leftrightarrow \begin{bmatrix} \star \\ \succ \end{bmatrix}, & \bar{\star} \leftrightarrow \begin{bmatrix} \star \\ \star \end{bmatrix}
\end{array}$$

Applying this bijection to the 49 relations for the ennea-algebra in [27], we see that the relations become the entries of the 7×7 matrix $\vec{\lambda}_T \,\square\, \vec{\lambda}_T$. $\square$

3.3.3. *Dendriform-Nijenhuis algebra.* The dendriform-Nijenhuis algebra [28] is equipped with 9 binary operations

$$\nearrow,\ \searrow,\ \swarrow,\ \nwarrow,\ \uparrow,\ \downarrow,\ \tilde{\prec},\ \tilde{\succ},\ \tilde{\bullet} \tag{9}$$

satisfying 28 relations.

Proposition 3.6. *The dendriform-Nijenhuis algebra is isomorphic to the ABQR algebra of type* $(\Omega_T, \Lambda_T) \,\square\, (\Omega_N, \Lambda_N)$ *where* (Ω_T, Λ_T) *is the type of dendriform trialgebra and* (Ω_N, Λ_N) *is the type of a NS-algebra in Ep. (4).*

Proof. We just give a correspondence between the binary operation of Leroux and ours. To distinguish operators in the dendriform trialgebra and NS-algebra, we denote $\Omega_N = \{<, >, \bullet\}$ and $\diamondsuit = < + > + \bullet$. A bijection between the binary operations in Eq. (9) and $\vec{\omega}_T \otimes \vec{\omega}_N$ is given by

$$\begin{array}{llll}
\nwarrow \leftrightarrow \begin{bmatrix} \prec \\ < \end{bmatrix}, & \uparrow \leftrightarrow \begin{bmatrix} \prec \\ \bullet \end{bmatrix}, & \nearrow \leftrightarrow \begin{bmatrix} \prec \\ > \end{bmatrix}, & \wedge \leftrightarrow \begin{bmatrix} \prec \\ \diamondsuit \end{bmatrix} \\
\tilde{\prec} \leftrightarrow \begin{bmatrix} \circ \\ < \end{bmatrix}, & \tilde{\bullet} \leftrightarrow \begin{bmatrix} \bullet \\ \circ \end{bmatrix}, & \tilde{\succ} \leftrightarrow \begin{bmatrix} \circ \\ > \end{bmatrix}, & \tilde{\star} \leftrightarrow \begin{bmatrix} \circ \\ \diamondsuit \end{bmatrix} \\
\swarrow \leftrightarrow \begin{bmatrix} \succ \\ < \end{bmatrix}, & \downarrow \leftrightarrow \begin{bmatrix} \succ \\ \bullet \end{bmatrix}, & \searrow \leftrightarrow \begin{bmatrix} \succ \\ > \end{bmatrix}, & \vee \leftrightarrow \begin{bmatrix} \succ \\ \diamondsuit \end{bmatrix} \\
\triangleleft \leftrightarrow \begin{bmatrix} \star \\ < \end{bmatrix}, & \bar{\bullet} \leftrightarrow \begin{bmatrix} \star \\ \bullet \end{bmatrix}, & \triangleright \leftrightarrow \begin{bmatrix} \star \\ > \end{bmatrix}, & \bar{\star} \leftrightarrow \begin{bmatrix} \star \\ \diamondsuit \end{bmatrix}
\end{array}$$

The operations on the fourth row and column are defined to be the sum along the corresponding projections. Then the relations in a dendriform-Nijenhuis algebra [28] is identified with the 7×4 matrix $\vec{\lambda}_T \,\square\, \vec{\lambda}_N$. $\square$

3.3.4. *Octo-algebra.* The octo-algebra of Leroux [29] is isomorphic to the product $(\Omega_Q \otimes \Omega_D, \Lambda_Q \,\square\, \Lambda_D)$ where (Ω_Q, Λ_Q) is the type of quadri-algebra and (Ω_D, Λ_D) is the type of dendriform dialgebra. As we will see later in §3.4, it is also the third power of dendriform dialgebra defined there. We will give details there (Proposition 3.8).

3.3.5. *Type M_1 and M_2 algebras.* Type M_1 and M_2 algebras were introduced in [29] as expansions of dipterous and anti-dipterous algebras. Let $(\Omega_i, \Lambda_i), i = 1, 2$, both be the type of the L-dipterous algebra. So we have $\Omega_i = \{\star_i, \succ_i\}$ with

$$\Lambda_i = \{(\star_i \otimes \star_i, \star_i \otimes \star_i), (\star_i \otimes \succ_i, \succ_i \otimes \succ_i), (\succ_i \otimes \star_i, \succ_i \otimes \star_i)\},\ i = 1, 2.$$

Then the M_2 algebra is isomorphic to the product $(\Omega_1 \otimes \Omega_2, \Lambda_1 \,\square\, \Lambda_2)$. The correspondence between the four binary operations of M_2 and $\vec{\omega}_1 \otimes \vec{\omega}_2$ is

$$\bullet_1 = \begin{bmatrix} \succ \\ \succ \end{bmatrix}, \bullet_2 = \begin{bmatrix} \succ \\ \star \end{bmatrix}, \bullet_3 = \begin{bmatrix} \star \\ \succ \end{bmatrix}, \bullet_4 = \begin{bmatrix} \star \\ \star \end{bmatrix}.$$

We skip the subscripts $i = 1, 2$, since it is clear from the context.

Similarly, The M_1 algebra in [29] is the product of the L-anti-dipterous algebra and L-dipterous algebra.

3.4. **Powers of an ABQR operad.** Inductively, we define the type product of any finite number of types of ABQR operads: given $(\Omega_i, \Lambda_i), 1 \leq i \leq n$, define

$$\mathop{\square}_{i=1}^{n} (\Omega_i, \Lambda_i) := \left(\mathop{\square}_{i=1}^{n-1} (\Omega_i, \Lambda_i) \right) \square\, (\Omega_n, \Lambda_n).$$

In particular, we define the powers of a specific type of ABQR operad. For simplicity, we only describe the powers of a dendriform trialgebra.

Definition 3.7. *A N-th power of the trialgebra is a* $\mathbf{k}$*-vector space D equipped with 3^N binary operations*

$$\begin{bmatrix} \odot_1 \\ \cdots \\ \odot_N \end{bmatrix},\ \odot_i \in \{\succ_i, \prec_i, \circ_i\},\ 1 \leq i \leq N$$

such that, for any choice of $(\odot_i^{(1)} \otimes \odot_i^{(2)}, \odot_i^{(3)} \otimes \odot_i^{(4)})$ in

$$\Lambda_i := \left\{ \begin{array}{l} (\prec_i \otimes \prec_i, \prec_i \otimes \star_i), (\succ_i \otimes \prec_i, \succ_i \otimes \prec_i), (\star_i \otimes \succ_i, \succ_i \otimes \succ_i), (\succ_i \otimes \circ_i, \succ_i \otimes \circ_i), \\ (\prec_i \otimes \circ_i, \circ_i \otimes \succ_i), (\circ_i \otimes \prec_i, \circ_i \otimes \prec_i), (\circ_i \otimes \circ_i, \circ_i \otimes \circ_i) \end{array} \right\}$$

as $1 \leq i \leq N$, we have

$$\left(x \begin{bmatrix} \odot_1^{(1)} \\ \cdots \\ \odot_N^{(1)} \end{bmatrix} y \right) \begin{bmatrix} \odot_1^{(2)} \\ \cdots \\ \odot_N^{(2)} \end{bmatrix} z = x \begin{bmatrix} \odot_1^{(3)} \\ \cdots \\ \odot_N^{(3)} \end{bmatrix} \left(y \begin{bmatrix} \odot_1^{(4)} \\ \cdots \\ \odot_N^{(4)} \end{bmatrix} z \right).$$

Here $\star_i = \succ_i + \prec_i + \circ_i$. When $\circ_i = 0$ for $1 \leq i \leq N$, we call the algebra the N-th power of the dendriform dialgebra.

Of course, the dendriform dialgebra and quadri-algebra (resp. dendriform trialgebra and ennea-algebra) are just the first and second power of dendriform dialgebra (resp. trialgebra). The octo-algebra introduced by Leroux [29] is the third power of the dendriform dialgebra (see below).

3.5. **Examples.** We now give examples of algebras with powers greater than two and more generally, of products with more than two factors.

3.5.1. *Octo-algebras.* The octo-algebra was introduced by Leroux [29]. It is defined using 8 operations

$$\nearrow_i, \nwarrow_i, \swarrow_i, \searrow_i\,,\ i = 1, 2$$

and 27 relations.

Proposition 3.8. *The octo-algebra is the third power of the dendriform dialgebra $(\vec{\omega}_D, \vec{\lambda}_D)$.*

Proof. The following table gives a correspondence between these 8 operations and the operations in $\vec{\omega}_D \otimes \vec{\omega}_D \otimes \vec{\omega}_D = \vec{\omega}_Q \otimes \vec{\omega}_D$ with $\vec{\omega}_Q$ the generators of the quadri-algebra. As in the case of the quadri-algebra, each operation on the third row and third column in the first two blocks is the sum of the corresponding projection, and each operation in the third block is the sum of the corresponding operations on the first and second blocks.

$$\begin{array}{ccc}
\nwarrow_1 \leftrightarrow \begin{bmatrix} \prec \\ \prec \\ \prec \end{bmatrix}, & \nearrow_1 \leftrightarrow \begin{bmatrix} \prec \\ \succ \\ \prec \end{bmatrix}, & \wedge_1 \leftrightarrow \begin{bmatrix} \prec \\ \star \\ \prec \end{bmatrix}, \\
\swarrow_1 \leftrightarrow \begin{bmatrix} \succ \\ \prec \\ \prec \end{bmatrix}, & \searrow_1 \leftrightarrow \begin{bmatrix} \succ \\ \succ \\ \prec \end{bmatrix}, & \vee_1 \leftrightarrow \begin{bmatrix} \succ \\ \star \\ \prec \end{bmatrix}, \\
\prec_1 \leftrightarrow \begin{bmatrix} \star \\ \prec \\ \prec \end{bmatrix}, & \succ_1 \leftrightarrow \begin{bmatrix} \star \\ \succ \\ \prec \end{bmatrix}, & \star_1 \leftrightarrow \begin{bmatrix} \star \\ \star \\ \prec \end{bmatrix}, \\
\hline
\nwarrow_2 \leftrightarrow \begin{bmatrix} \prec \\ \prec \\ \succ \end{bmatrix}, & \nearrow_2 \leftrightarrow \begin{bmatrix} \prec \\ \succ \\ \succ \end{bmatrix}, & \wedge_2 \leftrightarrow \begin{bmatrix} \prec \\ \star \\ \succ \end{bmatrix}, \\
\swarrow_2 \leftrightarrow \begin{bmatrix} \succ \\ \prec \\ \succ \end{bmatrix}, & \searrow_2 \leftrightarrow \begin{bmatrix} \succ \\ \succ \\ \succ \end{bmatrix}, & \vee_2 \leftrightarrow \begin{bmatrix} \succ \\ \star \\ \succ \end{bmatrix}, \\
\prec_2 \leftrightarrow \begin{bmatrix} \star \\ \prec \\ \succ \end{bmatrix}, & \succ_2 \leftrightarrow \begin{bmatrix} \star \\ \succ \\ \succ \end{bmatrix}, & \star_2 \leftrightarrow \begin{bmatrix} \star \\ \star \\ \succ \end{bmatrix}, \\
\hline
\nwarrow_{12} \leftrightarrow \begin{bmatrix} \prec \\ \prec \\ \star \end{bmatrix}, & \nearrow_{12} \leftrightarrow \begin{bmatrix} \prec \\ \succ \\ \star \end{bmatrix}, & \bigwedge_{12} \leftrightarrow \begin{bmatrix} \prec \\ \star \\ \star \end{bmatrix}, \\
\swarrow_{12} \leftrightarrow \begin{bmatrix} \succ \\ \prec \\ \star \end{bmatrix}, & \searrow_{12} \leftrightarrow \begin{bmatrix} \succ \\ \succ \\ \star \end{bmatrix}, & \bigvee_{12} \leftrightarrow \begin{bmatrix} \succ \\ \star \\ \star \end{bmatrix}, \\
\ll \leftrightarrow \begin{bmatrix} \star \\ \prec \\ \star \end{bmatrix}, & \gg \leftrightarrow \begin{bmatrix} \star \\ \succ \\ \star \end{bmatrix}, & \bar{\star} \leftrightarrow \begin{bmatrix} \star \\ \star \\ \star \end{bmatrix},
\end{array}$$

Then the 27 axioms of an octo-algebra in [29] correspond to entries in

$$\vec{\lambda}_D \,\square\, \vec{\lambda}_D \,\square\, \vec{\lambda}_D = \vec{\lambda}_Q \,\square\, \vec{\lambda}_D.$$

Here $\vec{\lambda}_Q$ is the relation vector of the quadri-algebra. $\square$

Recall that the opposite of the dialgebra is still a dialgebra. Considering also permutations of the three coordinates in $\begin{bmatrix} \odot_1 \\ \odot_2 \\ \odot_3 \end{bmatrix}$, we find that each element in the group of rigid motions of the cube gives an algebra isomorphic to the octo-algebra.

3.5.2. *The di-dipterous-anti-dipterous algebra.* It is easy to get new structures. For example, taking the product of a dendriform dialgebra, a L-dipterous algebra and a L-anti-dipterous algebra gives the di-dipterous-anti-dipterous algebra with 8 operations $\vec{\omega}_D \otimes \vec{\omega}_L \otimes \vec{\omega}_R$ and 27 relations $\vec{\lambda}_D \otimes \vec{\lambda}_L \otimes \vec{\lambda}_R$.

4. Duality of ABQR operads

We demonstrate how the description of operads in terms of their types can be used to give their duals. We will also give some examples.

4.1. **Definitions.** For an ABQR operad $\mathcal{P} = (\Omega, \Lambda)$, the dual operad is defined as follows. See [33, B.2] for further details.

Let $\check{\Omega} := \mathrm{Hom}(\Omega, \mathbf{k})$ be the dual space of Ω, giving the natural pairing

$$\langle\ ,\ \rangle_\Omega : \Omega \times \check{\Omega} \to \mathbf{k}.$$

Then via

$$\mathrm{Hom}(\Omega^{\otimes 2}, \mathbf{k}) \cong \mathrm{Hom}(\Omega, \check{\Omega}) \cong \mathrm{Hom}(\Omega, \mathbf{k}) \otimes \check{\Omega} \cong \check{\Omega}^{\otimes 2},$$

we get a natural (perfect) pairing

$$\langle\ ,\ \rangle_{\Omega^{\otimes 2}} : \Omega^{\otimes 2} \times \check{\Omega}^{\otimes 2} \to \mathbf{k}, \langle x \otimes y, a \otimes b\rangle_{\Omega^{\otimes 2}} = \langle x, a\rangle_\Omega\ \langle y, b\rangle_\Omega. \tag{10}$$

Further the isomorphism

$$\mathrm{Hom}(\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}, \mathbf{k}) \cong \mathrm{Hom}(\Omega^{\otimes 2}, \mathbf{k}) \oplus \mathrm{Hom}(\Omega^{\otimes 2}, \mathbf{k}) \cong \check{\Omega}^{\otimes 2} \oplus \check{\Omega}^{\otimes 2}$$

gives a perfect pairing

$$\langle\ ,\ \rangle_{2\Omega^{\otimes 2}} : (\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}) \times (\check{\Omega}^{\otimes 2} \oplus \check{\Omega}^{\otimes 2}) \to \mathbf{k} \tag{11}$$

by

$$\langle(\alpha, \beta), (\gamma, \delta)\rangle_{2\Omega^{\otimes 2}} = \langle\alpha, \gamma\rangle_{\Omega^{\otimes 2}} - \langle\beta, \delta\rangle_{\Omega^{\otimes 2}}, \alpha, \beta \in \Omega^{\otimes 2},\ \gamma, \delta \in \check{\Omega}^{\otimes 2}.$$

More precisely, let $\{x_i\}$ be a basis of Ω with dual basis $\{\check{x}_i\}$ in $\check{\Omega}$. Then

$$\langle(x_i \otimes x_j, x_k \otimes x_\ell), (\check{x}_s \otimes \check{x}_t, \check{x}_u \otimes \check{x}_v)\rangle_{2\Omega^{\otimes 2}} = \delta_{i,s}\delta_{j,t} - \delta_{k,u}\delta_{\ell,v}.$$

We now define $\Lambda^\perp$ to be the annihilator of $\Lambda \subseteq \Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ in $\check{\Omega}^{\otimes 2} \oplus \check{\Omega}^{\otimes 2}$ under the pairing $\langle\ ,\rangle_{2\Omega^{\otimes 2}}$. We call $\mathcal{P}^! := (\check{\Omega}, \Lambda^\perp)$ the **dual operad** of $\mathcal{P} = (\Omega, \Lambda)$ which is the Koszul dual in our special case. It follows from the definition that $(\mathcal{P}^!)^! = \mathcal{P}$.

4.2. **Examples.**

4.2.1. *Dual operad of the dendriform dialgebra.* This duality is given by Loday [33, Proposition 8.3]. Let (Ω_D, Λ_D) be the type of the operad for the dendriform dialgebra. Let $\{\dashv, \vdash\} \in \check{\Omega}_D$ be the dual basis of $\{\prec, \succ\}$ (in this order). Then $\Lambda_D^\perp$ is given by

$$\{(\dashv \otimes \dashv, \dashv \otimes \dashv), (\vdash \otimes \vdash, \vdash \otimes \vdash), (\dashv \otimes \dashv, \dashv \otimes \vdash), (\vdash \otimes \dashv, \vdash \otimes \dashv), (\dashv \otimes \vdash, \vdash \otimes \vdash)\}. \tag{12}$$

It is called the associative dialgebra $(\Omega_{AD}, \Lambda_{AD})$. There are many associative binary operations in Λ_{AD}, such as $\vdash$, $\dashv$ or their linear combinations.

4.2.2. *Dual operad of the dendriform trialgebra.* It is proved by Loday and Ronco [38, Theorem 3.1] that the dual operad of the dendriform trialgebra (Ω_T, Λ_T) is the associative trialgebra with three generators and 11 relations.

4.2.3. *Dual operad of the NS operad.* We now find the dual of the NS-algebra. The NS-algebra of Leroux [28] is defined with three binary operators $\vec{\omega}_N = \{\prec, \succ, \bullet\}$ that satisfy the relations

$$\vec{\lambda}_N = \Big\{(x \prec y) \prec z \overset{(1)}{=} x \prec (y \star z), (x \succ y) \prec z \overset{(2)}{=} x \succ (y \prec z),$$

$$(x \star y) \succ z \overset{(3)}{=} x \succ (y \succ z), (x \star y) \bullet z + (x \bullet y) \prec z \overset{(4)}{=} x \succ (y \bullet z) + x \bullet (y \star z)\Big\}$$

for $x, y, z \in D$. The labels on the equations are for later reference. Here $\star = \prec + \succ + \bullet$. Let $\{\dashv, \vdash, \circ\}$ be the basis of $\check{\Omega}_N$ that is dual to $\vec{\omega}_N$. We note that each term from $\vec{\omega}_N^{\otimes 2}$ occurs exactly once on the left hand side of the equations in $\vec{\lambda}_N$ and exactly once on the right hand side. Thus by inspection, we found that the following elements in $\check{\Omega}_N^{\otimes 2} \oplus \check{\Omega}_N^{\otimes 2}$ are perpendicular to $\vec{\lambda}_N$ and thus are in $\Lambda_N^\perp$.

$$\begin{aligned}
&(x \dashv y) \dashv z = x \dashv (y \dashv z), (x \dashv y) \dashv z = x \dashv (y \vdash z), (x \dashv y) \dashv z = x \dashv (x \circ z), \text{ (by (1))}\\
&(x \vdash y) \dashv z = x \vdash (y \dashv z), \text{ (by (2))}\\
&(x \vdash y) \vdash z = x \vdash (y \vdash z), (x \dashv y) \vdash z = x \vdash (y \vdash z), (x \circ y) \vdash z = x \vdash (y \vdash z), \text{ (by (3))}\\
&\theta = \eta, \text{ with } \theta \in \{(x \dashv y) \circ z, (x \vdash y) \circ z, (x \circ y) \circ z, (x \circ y) \dashv z\},\\
&\qquad \eta \in \{x \vdash (y \circ z), x \circ (y \dashv z), x \circ (y \vdash z), x \circ (y \circ z)\}, \text{ (by (4))}.
\end{aligned}$$

These are simplified (say by using Maple) to the following linearly independent subset.

$$\begin{aligned}
&(x \dashv y) \dashv z = x \dashv (y \dashv z), (x \dashv y) \dashv z = x \dashv (y \vdash z), (x \dashv y) \dashv z = x \dashv (x \circ z),\\
&(x \vdash y) \dashv z = x \vdash (y \dashv z),\\
&(x \vdash y) \vdash z = x \vdash (y \vdash z), (x \dashv y) \vdash z = x \vdash (y \vdash z), (x \circ y) \vdash z = x \vdash (y \vdash z),\\
&\theta = x \dashv (y \circ z) \text{ with } \theta \in \{(x \dashv y) \circ z, (x \vdash y) \circ z, (x \circ y) \circ z, (x \circ y) \dashv z\},\\
&(x \circ y) \dashv z = \eta \text{ with } \eta \in \{x \circ (y \dashv z), x \circ (y \vdash z), x \circ (y \circ z)\}.
\end{aligned}$$

Since it has 14 elements, $\vec{\lambda}_N$ has 4 elements and the dimension of $\Omega_N^{\otimes 2} \oplus \Omega_N^{\otimes 2}$ is 18, we see that this subset is a basis of $\Lambda_N^\perp$ and gives the relations of the dual operad of the NS operad. Of course, there are other choices for the basis. In particular, the relation $(x \circ y) \circ z = x \circ (y \circ z)$ is in $\vec{\lambda}_N^\perp$. We will call this the **associative Nijenhuis trialgebra**. It

is similar to the associative trialgebra in that all of the three binary operations $\{\dashv, \vdash, \circ\}$ are associative.

We remark that the products $\square$ and $\maltese$ are not related by taking the dual, in contrast to the duality between the products $\bullet$ and $\circ$ in [20]. For this we show that the dual of the quadri-algebra $(\Omega_Q, \Lambda_Q) = (\Omega_D, \Lambda_D) \,\square\, (\Omega_D, \Lambda_D)$ in Proposition 3.4 is not

$$(\Omega_D, \Lambda_D)^! \maltese (\Omega_D, \Lambda_D)^! = (\Omega_{AD}, \Lambda_{AD}) \maltese (\Omega_{AD}, \Lambda_{AD}).$$

By definition, the dual of (Ω_Q, Λ_Q) is given by $(\Omega_{AQ}, \Lambda_{AQ})$. Here Ω_{AQ} has basis

$$\left\{ \begin{bmatrix}\dashv\\ \dashv\end{bmatrix}, \begin{bmatrix}\dashv\\ \vdash\end{bmatrix}, \begin{bmatrix}\vdash\\ \dashv\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \right\}$$

which is dual to the basis

$$\left\{ \begin{bmatrix}\prec\\ \prec\end{bmatrix}, \begin{bmatrix}\prec\\ \succ\end{bmatrix}, \begin{bmatrix}\succ\\ \prec\end{bmatrix}, \begin{bmatrix}\succ\\ \succ\end{bmatrix} \right\}$$

of Ω_Q. Also $\Lambda_{AQ} = \Lambda_Q^{\perp}$. By Eq. (12), $(\vdash \otimes \dashv, \vdash \otimes \dashv)$ is in $\Lambda_D^{\perp} = \Lambda_{AD}$. However, $\left(\begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \vdash\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \dashv\end{bmatrix} \right)$ is not in $\Lambda_Q^{\perp}$. For example, $\left(\begin{bmatrix}\succ\\ \succ\end{bmatrix} \otimes \begin{bmatrix}\prec\\ \prec\end{bmatrix}, \begin{bmatrix}\succ\\ \succ\end{bmatrix} \otimes \begin{bmatrix}\prec\\ \prec\end{bmatrix} \right)$ is in Λ_Q. But by Eq. (10) and (11), we have

$$\begin{aligned}
&\left\langle \left(\begin{bmatrix}\succ\\ \succ\end{bmatrix} \otimes \begin{bmatrix}\prec\\ \prec\end{bmatrix}, \begin{bmatrix}\succ\\ \succ\end{bmatrix} \otimes \begin{bmatrix}\prec\\ \prec\end{bmatrix} \right), \left(\begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \vdash\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \dashv\end{bmatrix} \right) \right\rangle_{2\Omega_Q^{\otimes 2}} \\
= {} & \left\langle \begin{bmatrix}\succ\\ \succ\end{bmatrix} \otimes \begin{bmatrix}\prec\\ \prec\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \vdash\end{bmatrix} \right\rangle_{\Omega_Q^{\otimes 2}} - \left\langle \begin{bmatrix}\succ\\ \succ\end{bmatrix} \otimes \begin{bmatrix}\prec\\ \prec\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \dashv\end{bmatrix} \right\rangle_{\Omega_Q^{\otimes 2}} \\
= {} & \left\langle \begin{bmatrix}\succ\\ \succ\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \right\rangle_{\Omega_Q} \left\langle \begin{bmatrix}\prec\\ \prec\end{bmatrix}, \begin{bmatrix}\dashv\\ \vdash\end{bmatrix} \right\rangle_{\Omega_Q} - \left\langle \begin{bmatrix}\succ\\ \succ\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \right\rangle_{\Omega_Q} \left\langle \begin{bmatrix}\prec\\ \prec\end{bmatrix}, \begin{bmatrix}\dashv\\ \dashv\end{bmatrix} \right\rangle_{\Omega_Q} \\
= {} & -1
\end{aligned}$$

So $\left(\begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \vdash\end{bmatrix}, \begin{bmatrix}\vdash\\ \vdash\end{bmatrix} \otimes \begin{bmatrix}\dashv\\ \dashv\end{bmatrix} \right)$ is not in $\Lambda_Q^{\perp} = \Lambda_{AQ}$. Thus $\Lambda_{AD} \otimes \Omega_{AD}^{\otimes 2} \not\subseteq \Lambda_{AQ}$ and therefore $\Lambda_{AD} \maltese \Lambda_{AD} \not\subseteq \Lambda_{AQ}$.

5. ABQR algebras from linear operators

5.1. **Operators on ABQR algebras.** A common method used to obtain a new operad structure from a known operad structure is by means of a linear operator on the known operad. Such linear operators include (left, right and two-sided) Rota-Baxter operators and Nijenhuis operators. Examples of such constructions can be found in [4, 38, 27, 28, 29]. The constructions were usually verified by checking the relations for each case of operators and operads. We will verify the construction for (left, right and two-sided) Rota-Baxter and Nijenhuis operators on all types of ABQR algebras. We first give the definitions.

Definition 5.1. *Let D be an ABQR algebra of type (Ω, Λ). A linear operator P on D is called a* (*two-sided*) **Rota-Baxter operator of weight λ** (resp. **left Rota-Baxter**, resp. **right Rota-Baxter**) *if, for each $\odot \in \Omega$, we have*

$$P(x) \odot P(y) = P(P(x) \odot y + x \odot P(y) + \lambda x \odot y), \; x,\; y \in D$$

$$\text{(resp. } P(x)\odot P(y) = P(x\odot P(y)),\ x,\ y\in D) \tag{13}$$

$$\text{(resp. } P(x)\odot P(y) = P(P(x)\odot y),\ x,\ y\in D).$$

A linear operator N on D is called a **Nijenhuis operator** *if, for each $\odot\in\Omega$, we have*

$$N(x)\odot N(y) = N(N(x)\odot y + x\odot N(y) - N(x\odot y)),\ x,\ y\in D. \tag{14}$$

Theorem 5.2. *Let D be an ABQR algebra of type (Ω,Λ).*

(1) *Let P be a Rota-Baxter operator of weight λ on the ABQR algebra D. For each $\odot\in\Omega$, define binary operations on D by*

$$x\begin{bmatrix}\odot\\ \prec\end{bmatrix}y = x\odot P(y),\ x\begin{bmatrix}\odot\\ \succ\end{bmatrix}y = P(x)\odot y,\ x\begin{bmatrix}\odot\\ \circ\end{bmatrix}y = \lambda x\odot y.$$

Then these operations define an ABQR algebra on D of type $(\Omega\otimes\Omega_T, \Lambda\,\square\,\Lambda_T)$ where (Ω_T,Λ_T) is the type of dendriform trialgebra.

(2) *Let N be a Nijenhuis operator on the ABQR algebra D. For each $\odot\in\Omega$, define binary operations on D by*

$$x\begin{bmatrix}\odot\\ \prec\end{bmatrix}y = x\odot N(y),\ x\begin{bmatrix}\odot\\ \succ\end{bmatrix}y = N(x)\odot y,\ x\begin{bmatrix}\odot\\ \bullet\end{bmatrix}y = -N(x\odot y).$$

Then these operations define an ABQR algebra on D of type $(\Omega\otimes\Omega_N, \Lambda\,\square\,\Lambda_N)$ where (Ω_N,Λ_N) is the type of NS algebra in Eq. (4).

(3) *Let P be a left (resp. right) Rota-Baxter operator on the dendriform algebra D. For each $\odot\in\Omega$, define binary operations on D by*

$$x\begin{bmatrix}\odot\\ \succ\end{bmatrix}y = P(x)\odot y, x\begin{bmatrix}\odot\\ \vec{\star}\end{bmatrix}y = x\odot P(y),$$

$$\text{(resp. } x\begin{bmatrix}\odot\\ \prec\end{bmatrix}y = x\odot P(y), x\begin{bmatrix}\odot\\ \overleftarrow{\star}\end{bmatrix}y = P(x)\odot y).$$

Then these operations define an ABQR algebra on D of type $(\Omega\otimes\Omega_L, \Lambda\,\square\,\Lambda_L)$ ((resp. $(\Omega\otimes\Omega_R, \Lambda\,\square\,\Lambda_R)$) where (Ω_L,Λ_L) (resp. (Ω_R,Λ_R)) is the type of L-dipterous) (resp. L-anti-dipterous) algebra in Eq. (5) (resp. in Eq. (6)).

Proof. (1). Recall that

$$\begin{aligned}\vec{\lambda}_T = \ & \{(\prec\otimes\prec, \prec\otimes\star), (\succ\otimes\prec, \succ\otimes\prec), (\star\otimes\succ, \succ\otimes\succ), (\succ\otimes\circ, \succ\otimes\circ),\\ & (\prec\otimes\circ, \circ\otimes\succ), (\circ\otimes\prec, \circ\otimes\prec), (\circ\otimes\circ, \circ\otimes\circ)\}.\end{aligned}$$

To prove our claim we only need to prove that, for each

$$f = \sum_{j=1}^{r} (\odot_j^{(1)}\otimes\odot_j^{(2)}, \odot_j^{(3)}\otimes\odot_j^{(4)}) \in \Lambda$$

and each of the 7 pairs in $\vec{\lambda}_T$, the "tensor product" of the two is a relation on $\Omega\otimes\Omega_T$. We consider the 7 cases separately.

Case 1. The pair is $(\prec\otimes\prec, \prec\otimes\star)$. We need to verify that

$$f\,\square\,(\prec\otimes\prec,\prec\otimes\star)=\sum_{j=1}^{r}\left(\begin{bmatrix}\odot_j^{(1)}\\ \prec\end{bmatrix}\otimes\begin{bmatrix}\odot_j^{(2)}\\ \prec\end{bmatrix},\begin{bmatrix}\odot_j^{(3)}\\ \prec\end{bmatrix}\otimes\begin{bmatrix}\odot_j^{(4)}\\ \star\end{bmatrix}\right)$$

is a relation on D. We have

$$\begin{aligned}
\sum_{j=1}^{r}\left(x\begin{bmatrix}\odot_j^{(1)}\\ \prec\end{bmatrix}y\right)\begin{bmatrix}\odot_j^{(2)}\\ \prec\end{bmatrix}z &= \sum_{j=1}^{r}(x\odot_j^{(1)}P(y))\odot_j^{(2)}P(z)\\
&= \sum_{j=1}^{r}x\odot_j^{(3)}(P(y)\odot_j^{(4)}P(z))\\
&= \sum_{j=1}^{r}x\odot_j^{(3)}P(P(y)\odot_j^{(4)}z+y\odot_j^{(4)}P(z)+\lambda y\odot_j^{(4)}z)\\
&= \sum_{j=1}^{r}x\begin{bmatrix}\odot_j^{(3)}\\ \prec\end{bmatrix}(P(y)\odot_j^{(4)}z+y\odot_j^{(4)}P(z)+\lambda y\odot_j^{(4)}z)\\
&= \sum_{j=1}^{r}x\begin{bmatrix}\odot_j^{(3)}\\ \prec\end{bmatrix}\left(y\left(\begin{bmatrix}\odot_j^{(4)}\\ \prec\end{bmatrix}+\begin{bmatrix}\odot_j^{(4)}\\ \succ\end{bmatrix}+\begin{bmatrix}\odot_j^{(4)}\\ \circ\end{bmatrix}\right)z\right)\\
&= \sum_{j=1}^{r}x\begin{bmatrix}\odot_j^{(3)}\\ \prec\end{bmatrix}\left(y\begin{bmatrix}\odot_j^{(4)}\\ \star\end{bmatrix}z\right),
\end{aligned}$$

as is desired.

Case 2. The pair is $(\succ\otimes\prec, \succ\otimes\prec)$. Then we have

$$\begin{aligned}
\sum_{j=1}^{r}\left(x\begin{bmatrix}\odot_j^{(1)}\\ \succ\end{bmatrix}y\right)\begin{bmatrix}\odot_j^{(2)}\\ \prec\end{bmatrix}z &= \sum_{j=1}^{r}(P(x)\odot_j^{(1)}y)\odot_j^{(2)}P(z)\\
&= \sum_{j=1}^{r}P(x)\odot_j^{(3)}(y\odot_j^{(4)}P(z))\\
&= \sum_{j=1}^{r}x\begin{bmatrix}\odot_j^{(3)}\\ \succ\end{bmatrix}\left(y\begin{bmatrix}\odot_j^{(4)}\\ \prec\end{bmatrix}z\right).
\end{aligned}$$

Case 3. The pair is $(\star\otimes\succ, \succ\otimes\succ)$. We have

$$\begin{aligned}
\sum_{j=1}^{r}\left(x\begin{bmatrix}\odot_j^{(1)}\\ \star\end{bmatrix}y\right)\begin{bmatrix}\odot_j^{(2)}\\ \succ\end{bmatrix}z &= \sum_{j=1}^{r}P(x\odot_j^{(1)}P(y)+P(x)\odot_j^{(1)}y+\lambda x\odot_j^{(1)}y)\odot_j^{(2)}z\\
&= \sum_{j=1}^{r}(P(x)\odot_j^{(1)}P(y))\odot_j^{(2)}z
\end{aligned}$$

$$= \sum_{j=1}^{r} P(x) \odot_j^{(3)} (P(y) \odot_j^{(4)} z)$$
$$= \sum_{j=1}^{r} x \begin{bmatrix} \odot_j^{(3)} \\ \succ \end{bmatrix} \left(y \begin{bmatrix} \odot_j^{(4)} \\ \succ \end{bmatrix} z \right).$$

Case 4. The pair is $(\succ \otimes \circ, \succ \otimes \circ)$. We have

$$\sum_{j=1}^{r} \left(x \begin{bmatrix} \odot_j^{(1)} \\ \succ \end{bmatrix} y \right) \begin{bmatrix} \odot_j^{(2)} \\ \circ \end{bmatrix} z$$
$$= \sum_{j=1}^{r} (P(x) \odot_j^{(1)} y) \odot_j^{(2)} \lambda z$$
$$= \sum_{j=1}^{r} P(x) \odot_j^{(3)} (y \odot_j^{(4)} \lambda z)$$
$$= \sum_{j=1}^{r} x \begin{bmatrix} \odot_j^{(3)} \\ \succ \end{bmatrix} \left(y \begin{bmatrix} \odot_j^{(4)} \\ \circ \end{bmatrix} z \right).$$

The cases for the pairs $(\prec \otimes \circ, \circ \otimes \succ)$, $(\circ \otimes \prec, \circ \otimes \prec)$ and $(\circ \otimes \circ, \circ \otimes \circ)$ are proved in the same way as Case 4.

2. Now we consider a Nijenhuis operator N. The relation set of the NS-algebra is

$$\begin{aligned} \vec{\lambda}_N = \quad & \{(\prec \otimes \prec, \prec \otimes \star), (\succ \otimes \prec, \succ \otimes \prec), (\star \otimes \succ, \succ \otimes \succ), \\ & (\star \otimes \bullet + \bullet \otimes \prec, \succ \otimes \bullet + \bullet \otimes \star)\}. \end{aligned}$$

So we only need to prove that, for each

$$f = \sum_{j=1}^{r} (\odot_j^{(1)} \otimes \odot_j^{(2)}, \odot_j^{(3)} \otimes \odot_j^{(4)}) \in \Lambda$$

and each of the 4 pairs in $\vec{\lambda}_N$, the "tensor product" of the two is a relation for D. The verification of the first three pairs is the same as the first three cases in the Rota-Baxter operator case. For the fourth case, taking f as above, we have

$$\sum_{j=1}^{r} \left(\left(x \begin{bmatrix} \odot_j^{(1)} \\ \star \end{bmatrix} y \right) \begin{bmatrix} \odot_j^{(2)} \\ \bullet \end{bmatrix} z + \left(x \begin{bmatrix} \odot_j^{(1)} \\ \bullet \end{bmatrix} y \right) \begin{bmatrix} \odot_j^{(2)} \\ \prec \end{bmatrix} z \right)$$
$$= \sum_{j=1}^{r} \Big(-N((N(x) \odot_j^{(1)} y + \odot_j^{(1)} N(y) - N(x \odot_j^{(1)} y)) \odot_j^{(2)} z) - N(x \odot_j^{(1)} y) \odot_j^{(2)} N(z) \Big)$$
$$= \sum_{j=1}^{r} \Big(- N((N(x) \odot_j^{(1)} y + x \odot_j^{(1)} N(y) - N(x \odot_j^{(1)} y)) \odot_j^{(2)} z)$$

$$
\begin{aligned}
& -N(N(x \odot_j^{(1)} y) \odot_j^{(2)} z + (x \odot_j^{(1)} y) \odot_j^{(2)} N(z) - N((x \odot_j^{(1)} y) \odot_j^{(2)} z))\Big) \\
= \ & \sum_{j=1}^{r} \Big(- N((N(x) \odot_j^{(1)} y) \odot_j^{(2)} z + (x \odot_j^{(1)} N(y)) \odot_j^{(2)} z \\
& +(x \odot_j^{(1)} y) \odot_j^{(2)} N(z) - N((x \odot_j^{(1)} y) \odot_j^{(2)} z))\Big) \\
= \ & -N\left(\sum_{j=1}^{r}(N(x) \odot_j^{(1)} y) \odot_j^{(2)} z + \sum_{j=1}^{r}(x \odot_j^{(1)} N(y)) \odot_j^{(2)} z \right.\\
& \left. + \sum_{j=1}^{r} (x \odot_j^{(1)} y) \odot_j^{(2)} N(z) - N\left(\sum_{j=1}^{r}(x \odot_j^{(1)} y) \odot_j^{(2)} z\right)\right) \\
= \ & -N\left(\sum_{j=1}^{r} N(x) \odot_j^{(3)} (y \odot_j^{(4)} z) + \sum_{j=1}^{r} x \odot_j^{(3)} (N(y) \odot_j^{(4)} z) \right.\\
& \left. + \sum_{j=1}^{r} x \odot_j^{(3)} (y \odot_j^{(4)} N(z)) - N\left(\sum_{j=1}^{r} x \odot_j^{(3)} (y \odot_j^{(4)} z)\right)\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \sum_{j=1}^{r} \left(x \begin{bmatrix} \odot_j^{(3)} \\ \succ \end{bmatrix} \left(y \begin{bmatrix} \odot_j^{(4)} \\ \bullet \end{bmatrix} z \right) + x \begin{bmatrix} \odot_j^{(3)} \\ \bullet \end{bmatrix} \left(y \begin{bmatrix} \odot_j^{(4)} \\ \star \end{bmatrix} z \right) \right) \\
= \ & \sum_{j=1}^{r}\Big(-N(x) \odot_j^{(3)} N(y \odot_j^{(4)} z) - N(x \odot_j^{(3)} (N(y) \odot_j^{(4)} z + y \odot_j^{(4)} N(z) - N(y \odot_j^{(4)} z)))\Big) \\
= \ & \sum_{j=1}^{r} \Big(- N(N(x) \odot_j^{(3)} (y \odot_j^{(4)} z) + x \odot_j^{(3)} N(y \odot_j^{(4)} z) - N(x \odot_j^{(3)} (y \odot_j^{(4)} z))) \\
& -N(x \odot_j^{(3)} (N(y) \odot_j^{(4)} z + y \odot_j^{(4)} N(z) - N(y \odot_j^{(4)} z)))\Big) \\
= \ & \sum_{j=1}^{r} -N(N(x) \odot_j^{(3)} (y \odot_j^{(4)} z) - N(x \odot_j^{(3)} (y \odot_j^{(4)} z)) \\
& +x \odot_j^{(3)} (N(y) \odot_j^{(4)} z) + x \odot_j^{(3)} (y \odot_j^{(4)} N(z))).
\end{aligned}
$$

This verifies the last relation.

The proof of (3) is the same as (actually simpler than) the proof of (1). $\square$

5.2. **Algebras with commuting operators.** Using Theorem 5.2 inductively, we get

Corollary 5.3. *Let $(D, \{P_i\}_i)$ be an algebra of type (Ω, Λ) with k commuting linear operators P_i, such as Rota-Baxter, Nijenhuis, left Rota-Baxter or right Rota-Baxter operators. We obtain on D an algebra structure of type $(\Omega, \Lambda) \,\square\, (\,\square_i(\Omega_i, \Lambda_i))$ where (Ω_i, Λ_i) is the algebra type corresponding to the operator P_i, $1 \le i \le k$.*

5.3. **Examples.** Some of the previous known constructions can be obtained as special cases of Theorem 5.2 and Corollary 5.3. They can be found

(1) in [1] where a Rota-Baxter operator of weight zero on an associative algebra is used to construct a dendriform dialgebra;
(2) in [10, 27] where a Rota-Baxter operator of non-zero weight on an associative algebra is used to construct a dendriform trialgebra;
(3) in [4] where a Rota-Baxter operator of weight zero on a dendriform dialgebra is used to construct a dendriform quadri-algebra;
(4) in [4] where a pair of commuting Rota-Baxter operator of weight zero on an associative algebra is used to construct a dendriform quadri-algebra;
(5) in [27] where a Rota-Baxter operator of non-zero weight on a dendriform trialgebra is used to construct an ennea-algebra;
(6) in [27] where a pair of commuting Rota-Baxter operators of non-zero weight on an associative algebra is used to construct an ennea-algebra;
(7) in [28] where a Nijenhuis operator on an associative algebra is used to construct a NS-algebra;
(8) in [28] where a Nijenhuis operator on a dendriform trialgebra is used to construct a dendriform-Nijenhuis algebra.
(9) in [29] where a left (resp. right) Rota-Baxter operator on an associative algebra is used to construct a L-dipterous (an anti-L-dipterous) algebra;
(10) in [29] where a commuting pair of a right and a left Rota-Baxter operators (resp. of two left Rota-Baxter operators) on an associative algebra is used to construct an M_1 algebra (resp. an M_2 algebra);
(11) in [29] where a set of three pairwise commuting Rota-Baxter operators of weight zero on an associative algebra is used construct an octo-algebra.

We end this paper with two more examples. It is easy to verify that if P is a Rota-Baxter operator of weight $\lambda \in \mathbf{k}$ on an ABQR algebra D of type (Ω, Λ), i.e. P satisfies relation (13), then the operator $\tilde{P} := -\lambda\,\mathrm{id}_D - P$ also is a Rota-Baxter operator of weight λ on D. Thus D has two commuting Rota-Baxter operators P and $\tilde{P}$ of weight λ. So by Corollary 5.3, D is equipped with an ABQR algebra of type

$$(\Omega, \Lambda) \,\Box\, (\Omega_T, \Lambda_T) \,\Box\, (\Omega_T, \Lambda_T) = (\Omega, \Lambda) \,\Box\, (\Omega_E, \Lambda_E).$$

Here (Ω_T, Λ_T) is the type of dendriform trialgebra and (Ω_E, Λ_E) is the type of dendriform ennea-algebra.

Similarly, if N is a Nijenhuis operator on D, i.e. it satisfies relation (14), then so is the operator $\tilde{N} = id_D - N$. Thus D is equipped with an ABQR algebra of type $(\Omega, \Lambda) \,\Box\, (\Omega_N, \Lambda_N) \,\Box\, (\Omega_N, \Lambda_N)$. Here (Ω_N, Λ_N) is the type of NS algebra.

Returning to the case when P is a Rota-Baxter operator, define for each $\odot \in \Omega$, binary operations on D by

$$x \begin{bmatrix} \odot \\ \prec \end{bmatrix} y = x \odot P(y),\ x \begin{bmatrix} \odot \\ \succ \end{bmatrix} y = -\tilde{P}(x) \odot y.$$

Then these operations define an ABQR algebra on D of type $(\Omega \otimes \Omega_D, \Lambda \square \Lambda_D)$ where (Ω_D, Λ_D) is the type of dendriform dialgebra. The proof is the same as the one given in [10].

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